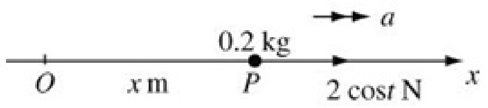
Exercise 3A

1



a $F = ma$

$2\cos t = 0.2a$

$0.2\frac{dv}{dt} = 2\cos t$ ← Force is a function of time so use $a = \frac{dv}{dt}$.

$v = \frac{2}{0.2}\int \cos t \, dt$ ← Integrate to obtain an expression for v.

$v = 10\sin t + c$ ← Don't forget the constant.

$t = 0 \; v = 0$

$0 = 0 + c \quad \therefore c = 0$

$v = 10\sin t$

When $t = 2$ gives $v = 10\sin 2 = 9.092...$

When $t = 2$ the speed of P is 9.09 m s^{-1} (3 s.f.)

b $t = 3$ gives $v = 10\sin 3 = 1.411...$

When $t = 3$ the speed of P is 1.41 m s^{-1} (3 s.f.)

c $v = 0$ gives $0 = 10\sin t$ ← P comes to rest when $v = 0$.

$\sin t = 0$

$t = 0, \pi, ...$

P first comes to rest when $t = \pi$ s. ← Exact answers are best.

d $v = 10\sin t$

$\frac{dx}{dt} = 10\sin t$

$x = 10\int \sin t \, dt$ ← Integrate to obtain an expression for x.

$x = -10\cos t + K$

$t = 0, x = 0 \quad 0 = -10 + K \therefore K = 10$

$x = -10\cos t + 10$

$t = 2$ gives $x = -10\cos t + 10 = 14.16...$

When $t = 2$ $OP = 14.2$ m (3 s.f.)

e $t = \pi$ gives $x = -10\cos\pi + 10$

$= 10 + 10 = 20$

When P comes to rest $OP = 20$ m.

2 a $F = ma$

$$\frac{60\,000}{(t+5)^2} = 1200a$$

$$a = \frac{50}{(t+5)^2}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{50}{(t+5)^2}$$

Force is a function of time so use $a = \frac{\mathrm{d}v}{\mathrm{d}t}$.

$$v = \int \frac{50}{(t+5)^2}\,\mathrm{d}t$$

Integrate to obtain an expression for v.

$$v = -\frac{50}{(t+5)} + c$$

$$t = 0,\ v = 0 \quad \therefore 0 = -\frac{50}{5} + c$$

$$c = 10$$

$$v = -\frac{50}{t+5} + 10$$

$$\text{As } t \to \infty - \frac{50}{t+5} + 0$$

$$\therefore V = 10$$

b $v = -\frac{50}{t+5} + 10$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{50}{t+5} + 10$$

$$x = -50\ln(t+5) + 10t + K$$

$$t = 0, x = 0 \qquad 0 = -50\ln 5 + K$$

$$K = 50\ln 5$$

$$\therefore x = -50\ln(t+5) + 10t + 50\ln 5$$

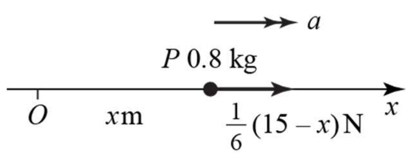
$$t = 4 \quad x = -50\ln 9 + 40 + 50\ln 5$$

$$x = 40 + 50\ln\frac{5}{9}$$

$$x = 10.61\ldots$$

The van moves 10.6 m in the first 4 seconds (3 s.f.)

3



a Maximum speed $\Rightarrow$ acceleration zero $\Rightarrow$ force is zero

$$\therefore \frac{1}{6}(15 - x) = 0 \quad \therefore x = 15$$

3 b $F = ma$

$$\frac{1}{6}(15-x) = 0.8a$$

$$a = \frac{1}{4.8}(15-x)$$

$$v\frac{dv}{dx} = \frac{1}{4.8}(15-x)$$

Force is a function of x so use $a = v\frac{dv}{dx}$.

$$\int v\,dv = \frac{1}{4.8}\int (15-x)\,dx$$

Separate the variables.

$$\frac{1}{2}v^2 = \frac{1}{4.8}\left(15x - \frac{1}{2}x^2\right) + c$$

$x = 15, v = 12$

a tells you the initial conditions.

$$\frac{1}{2}\times 12^2 = \frac{1}{4.8}\left(15\times 15 - \frac{1}{2}\times 15^2\right) + c$$

$$c = \frac{1}{2}\times 12^2 - \frac{1}{4.8}\times\frac{1}{2}\times 15^2$$

$$c = 48.5625$$

$$\frac{1}{2}v^2 = \frac{1}{4.8}\left(15x - \frac{1}{2}x^2\right) + 48.5625$$

$t = 0, x = 0 \quad v^2 = 2\times 48.5625$

$v = 9.855$

P is at 0 when $t = 0$.

When $t = 0$ P's speed is 9.86 m s^{-1} (3 s.f.)

4 a $0.75v\frac{dv}{dx} = 2e^{-x} + 2$

$$v\frac{dv}{dx} = \frac{8}{3}e^{-x} + \frac{8}{3}$$

Separating the variables and integrating:

$$\int v\,dv = \int\left(\frac{8}{3}e^{-x} + \frac{8}{3}\right)dx + c$$

$$\frac{v^2}{2} = -\frac{8}{3}e^{-x} + \frac{8x}{3} + c$$

When $x = 0,\ v = 5$:

$$\frac{5^2}{2} = -\frac{8}{3}e^{-0} + \frac{8\times 0}{3} + c$$

$$c = \frac{91}{6}$$

$$\frac{v^2}{2} = -\frac{8}{3}e^{-x} + \frac{8x}{3} + \frac{91}{6}$$

When $x = 3$

$$\frac{v^2}{2} = -\frac{8}{3}e^{-3} + 8 + \frac{91}{6}$$

$\Rightarrow v = 6.79$ m s^{-1}

4 **b** When $x = 7$

$$\frac{v^2}{2} = -\frac{8}{3}e^{-7} + \frac{56}{3} + \frac{91}{6}$$

$$\Rightarrow v = 8.23 \text{ m s}^{-1}$$

c Work done $= \int_3^7 F\,dx$

$$= \int_3^7 \left(2e^{-x} + 2\right) dx$$

$$= \left[-2e^{-x} + 2x\right]_3^7$$

$$= -2e^{-7} + 14 - \left(-2e^{-3} + 6\right)$$

$$= 8.10 \text{ J}$$

5 $F = mv\frac{dv}{dx}$

$$\frac{3}{x+2} = \frac{1}{2}v\frac{dv}{dx}$$

Separating the variables and integrating:

$$\int_0^X \frac{3}{x+2}dx = \int_{1.5}^2 \frac{v}{2}dv$$

$$3\ln(x+2) = \frac{v^2}{4} + c$$

$$t = 0,\ v = 1.5 \quad \therefore\ 3\ln(2) = \frac{1.5^2}{4} + c$$

$$c = 1.517...$$

When $v = 2$

$$\ln(x+2) = \frac{2^2}{12} + \frac{1.517}{3} = 0.83898...$$

$$x = e^{0.83898} - 2$$

$$x = 0.314$$

6 a $m\frac{\mathrm{d}v}{\mathrm{d}t} = F$

$\frac{1}{4}\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{8}{(t+1)^2}$

$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{32}{(t+1)^2}$

Integrating gives:

$v = \frac{32}{t+1} + c$

At $t = 0,\ v = 10$

$10 = 32 + c$

$c = -22$

$v = \frac{32}{t+1} - 22$

$v = 2\left(\frac{16}{(t+1)} - 11\right)$

b $x = \int v\ \mathrm{d}t = \int\left(\frac{32}{t+1} - 22\right)\mathrm{d}t$

$x = 32\ln(t+1) - 22t + c$

At $t = 0,\ x = 0$, so $c = 0$

$x = 32\ln(t+1) - 22t$

When $t = 5$:

$x = 32\ln 6 - 22 \times 5$

$x = 32\ln 6 - 110$

7 F is directed towards O, so

$$F = -\frac{k}{(x+2)^2}$$

$$mv\frac{dv}{dx} = -\frac{k}{(x+2)^2}$$

$$\frac{3}{5}v\frac{dv}{dx} = -\frac{k}{(x+2)^2}$$

Separating the variables and integrating:

$$\int \frac{3v}{5}\,dv = -\int \frac{k}{(x+2)^2}$$

$$\frac{3v^2}{10} = \frac{k}{(x+2)} + c$$

At $x = 3$, $v^2 = 25$:

$$7.5 = \frac{k}{5} + c$$

$$37.5 = k + 5c \qquad \textbf{(1)}$$

At $x = 8$, $v^2 = 3$:

$$0.9 = \frac{k}{10} + c$$

$$9 = k + 10c \qquad \textbf{(2)}$$

Solving equations **(1)** and **(2)** simultaneously gives:
$5c = -28.5$
So $c = -5.7$

Therefore $k = 66$

Challenge

a Work done $= \int_a^b \left(3x^2 - x^{\frac{1}{3}}\right) \mathrm{d}x$

$$= \left[x^3 - \frac{3}{4}x^{\frac{4}{3}}\right]_a^b$$

$$= b^3 - \frac{3}{4}b^{\frac{4}{3}} - a^3 + \frac{3}{4}a^{\frac{4}{3}}$$

Hence the work done is independent of the initial velocity.

b Work done $= \int_0^6 \left(3x^2 - x^{\frac{1}{3}}\right) \mathrm{d}x$

$$= \left[x^3 - \frac{3}{4}x^{\frac{4}{3}}\right]_0^6$$

$$= 6^3 - \frac{3}{4} \times 6^{\frac{4}{3}}$$

$$= 208 \text{ J}$$